Elastic Wave Equation

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1 Seismic Wave Propagation

1.1 Mathematical introduction

- https://wiki.seg.org/wiki/Mathematical_foundation_of_elastic_wave_propagation
- https://igppweb.ucsd.edu/~guy/sio227a/ch3.pdf

1.1.1 Wave Equation

Remember the wave equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = c^2 \Delta u$$

where - $u(x, y, t)$ is the displacement vector of a particle in (x, y) at moment t - $c(x, y)$ is the wave velocity in (x, y)

The Seismic Wave Equation has a lot in common with this equation. The main difference comes from the fact that when the wave propagates itself, if the propagation medium is elastic, it gets deformed. We present the case of the *Elastic Wave Equation* also called the *Seismic Wave Equation*.

In short, the *Seismic Wave Equation* is obtained by expressing the *stress tensor* appearing in the *Momentum Equation* in terms of the *linear and isotropic stress-strain relationship*. We develop those notions.

1.1.2 The Momentum Equation

On an infinitesimal cube in the x_1, x_2, x_3 coordinates system, the force (tension $\times$ surface) on one of the surfaces $\hat{x_1}$ is given by:

$$F(\hat{x_1}) = t(\hat{x_1}) dx_2 dx_3 = \tau \hat{x_1} dx_2 dx_3 = (\tau_{11}, \tau_{21}, \tau_{31})^T dx_2 dx_3$$

Note that, for such force to exist, spatial gradients must exist in the stress field:

$$F(\hat{x_1}) = \frac{\mathrm{d}\tau_1}{\mathrm{d}x_1} dx_1 dx_2 dx_3$$

which can be transformed by the index notation to the toal expression:

$$F_i = \sum_{j=1}^{3} \frac{\mathrm{d}\tau_{ij}}{\mathrm{d}x_j} d_1 dx_2 dx_3 = \mathrm{d}_j \tau_{ij} dx_1 dx_2 dx_3$$

where the term $\mathrm{d}_j \tau_{ij}$ is the divergence of the stress tensor.

Note also that there can be an external body force (eg. gravity) that depends on the mass of the body:

$$F_i^{body} = f_i dx_1 dx_2 dx_3$$

where the mass depends on the density ρ as:

$$m = \rho dx_1 dx_2 dx_3$$

which leads to a total force given by:

$$F_i^{total} = \mathrm{d}_j \tau_{ij} dx_1 dx_2 dx_3 + f_i dx_1 dx_2 dx_3$$

Now, recall that $F = ma$, and that the accelaration is the second derivative of the displacement:

$$a = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$

We can therefore substitute F, m and a to obtain:

$$\mathrm{d}_j \tau_{ij} dx_1 dx_2 dx_3 + f_i dx_1 dx_2 dx_3 = \rho dx_1 dx_2 dx_3 \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$

Remark that the factors $dx_1 dx_2 dx_3$ can be cancelled (since they are non zero) to get:

$$\mathrm{d}_j \tau_{ij} + f_i = \rho \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$

Often, the external forces will be neglected (but this is an hypothese), to get the homogeneous equation:

$$\mathrm{d}_j \tau_{ij} = \rho \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$

This equation is called the *Homogeneous Momentum Equation*, or *Homogeneous Equation of Motions*. Since it relies on the stress tensor τ_{ij}, we can constrain this term for specific cases. Here, we will consider the *linear and isotropic stress-strain relationship*.

1.1.3 The Seismic Wave Equation

The *linear and isotropic stress-strain relationship* is given by:

$$\tau_{ij} = \lambda \delta ij e_{kk} + 2\mu e_{ij}$$

where - λ and μ are called the *Lamé parameters* and depends on the propagation medium composition - δ_{ij} is the Kronecker delta

Since $e_{ij} = \frac{1}{2}(\mathrm{d}_i u_j + \mathrm{d}_j u_i)$, we can rewrite the relationship as:

$$\tau_{ij} = \lambda \delta_{ij} \mathrm{d}_k u_k + \mu(\mathrm{d}_i u_j + \mathrm{d}_j u_i)$$

Now, if we substitute this formulation in the *Homogeneous Momentum Equation*, we get (note the index notation):

$$\rho \frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} = \mathrm{d}_j[\lambda \delta_{ij} \mathrm{d}_k u_k + \mu(\mathrm{d}_i u_j + \mathrm{d}_j u_i)]$$

which can be rewritten as:

$$\rho \frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} = \mathrm{d}_i \lambda \mathrm{d}_k u_k + \mathrm{d}_j \mu(\mathrm{d}_i u_j + \mathrm{d}_j u_i) + \lambda \mathrm{d}_i \mathrm{d}_k u_k + \mu \mathrm{d}_i \mathrm{d}_j u_j + \mu \mathrm{d}_j d_j u_i$$

We write this in vector notation, using $\ddot{u} = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$:

$$\rho \ddot{u} = \nabla \lambda \nabla . u + \nabla \mu (\nabla u + (\nabla u)^T) + (\lambda + \mu) \nabla \nabla . u + \mu \nabla^2 u$$

where - ∇u is the gradient of u - $\nabla . u$ is the divergence of u - $\nabla^2 u$ is the laplacian of u

This equation is called the *Seismic Wave Equation*. If we neglect the external forces terms, we get:

$$\rho \ddot{u} = (\lambda + \mu) \nabla \nabla . u + \mu \nabla^2 u$$

1.1.4 P and S Wave decomposition

Using the *Helmoltz Decomposition*, also called the *fundamental theorem of vector calculus*, which states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be decomposed into the sum of a curl-free and divergence-free vector field.

We can derive the *Seismic Wave Equation* by fixing two variables:

- The P Wave velocity α such that $\alpha^2 = \frac{\lambda+2\mu}{\rho}$
- The S Wave velocity β such that $\beta^2 = \frac{\mu}{\rho}$

The *Seismic Wave Equation* can then be rewritten as:

$$\ddot{u} = \alpha^2 \nabla\nabla.u - \beta^2 \nabla \times \nabla \times u$$

with $\nabla \times u$ being the curl operator, or, for dimensions where the curl operator is not defined:

$$\ddot{u} = \alpha^2 \nabla\nabla.u - \beta^2 (\nabla\nabla.u - \nabla^2 u)$$

1.1.5 Units

- u: cm
- ρ: $\frac{kg}{cm^3}$
- λ and μ: $\frac{N}{cm^2}$